On geometry of the scator space

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Abstract

We consider the scator space - a hypercomplex, non-distributive hyperbolic algebra introduced by Fernández-Guasti and Zaldívar. We discuss isometries of the scator space and find consequent method for treating them algebraically, along with scators themselves. It occurs that introduction of zero divisors cannot be avoided while dealing with these isometries. The scator algebra may be endowed with a nice physical interpretation, although it suffers from lack of some physically demanded important features. Despite that, there arises some open questions, e.g., whether hypothetical tachyons can be considered as usual particles possessing time-like trajectories.

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1 Introduction

Following Fernández-Guasti and Zaldívar [1] we consider a commutative, non-distributive 1+2 dimensional algebra S, which is also associative provided that divisors of zero are excluded. The elements of this algebra will be called 1+2 dimensional scators [1]. Scators (a kind of hypercomplex numbers) are denoted by $\overset{o}{a} = (a_0; a_1, a_2)$, where components a_1, a_2 are referred to as director components, and a_0 is usually called scalar component, or, in

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physical context, temporal component. The space of scators possesses the additive structure of usual vector space, with scalars being its subset, closed under addition and multiplication. In this paper we confine ourselves to this definition of a scator, leaving aside earlier concepts of scators as objects generalizing scalars and vectors, see [2, 3].

It was shown in [4] that this algebra may be given physical interpretation corresponding to some ideas of special relativity, although metric in the scator space is different from the standard metric of Minkowski space. This scator metric (defined below) is called scator-deformed Lorentz metric. It was emphasized that scators and deformed metrics can describe a kinematics of some kind of particles, including hypothetical tachyons.

In our paper we study metric properties of scators, paying special attention to proper definitions for causal realms appearing in this framework, what finally will lead to some convergence with work of Kapuścik [5].

We emphasize the fact that multiplication acts in a non-distributive way, what is the hallmark of the structure. Indeed, we have

$$\overset{\circ}{ab} = (a_0; a_1, a_2)(b_0; b_1, b_2) =
= \left(a_0b_0 + a_1b_1 + a_2b_2 + \frac{a_1a_2b_1b_2}{a_0b_0}; a_0b_1 + a_1b_0 + \frac{a_1a_2b_2}{a_0} + \frac{a_2b_1b_2}{b_0},
a_0b_2 + a_2b_0 + \frac{a_1a_2b_1}{a_0} + \frac{a_1b_1b_2}{b_0}\right).$$
(1.1)

Therefore, computing

$$\Delta(\overset{\circ}{a},\overset{\circ}{b};\overset{\circ}{c}) := (\overset{\circ}{a} + \overset{\circ}{b})\overset{\circ}{c} - \overset{\circ}{ac} - \overset{\circ}{bc} = \frac{(b_0a_1 - a_0b_1)(a_0b_2 - b_0a_2)}{a_0b_0(a_0 + b_0)} \left(\frac{c_1c_2}{c_0};c_2,c_1\right),$$
(1.2)

where $\stackrel{o}{c} = (c_0; c_1, c_2)$, we easily see that in the generic case the righ-hand side does not vanish, i.e., $(\stackrel{o}{a} + \stackrel{o}{b})\stackrel{o}{c} \neq \stackrel{o}{a}\stackrel{o}{c} + \stackrel{o}{b}\stackrel{o}{c}$ provided that $b_0a_1 \neq a_0b_1$ and $a_0b_2 \neq b_0a_2$. Hence, in general, the scator product is not distributive. By the way, the scator appearing on the right-hand side of (1.2) will be referred to as dual to $\stackrel{o}{c}$ (see Definition 2.1 below).

Many properties of the scator product (1.1) were widely investigated in many contexts [1, 8, 9], also physical [4]. To gain some insight in possible physical interpretation of the scator algebra, we recall here some basic terminology from [1] and [9], although modified a little:

Definition 1.1. The modulus squared of a scator is given by

$$\|\overset{o}{a}\|^2 = \overset{o \, o \, *}{aa} = a_0^2 \left(1 - \frac{a_1^2}{a_0^2} \right) \left(1 - \frac{a_2^2}{a_0^2} \right) = a_0^2 - a_1^2 - a_2^2 + \frac{a_1^2 a_2^2}{a_0^2}. \tag{1.3}$$

Definition 1.2. We say that a scator is time-like, if

$$a_0^2 > a_1^2$$
, and $a_0^2 > a_2^2$, or $a_0^2 < a_2^2$, and $a_0^2 < a_1^2$, (1.4)

it is said to be space-like, if

$$a_0^2 < a_1^2$$
, and $a_0^2 > a_2^2$ or $a_0^2 < a_2^2$, and $a_0^2 > a_1^2$, (1.5)

and it is light-like, when

$$a_0^2 = a_1^2, \quad or \quad a_0^2 = a_2^2.$$
 (1.6)

The division proposed above is analogous to what is well known from special relativity. Note that in both cases componentwise addition of scators or vectors does not preserve this division. On the other hand, the norm of a product of two scators is just a product of their norms [9], which is very useful in this context. In this paper we present new results concerning the metric structure of the scator space, extending results of [4, 9].

For instance, we indicate that bipyramid considered in [4] has some kind of time-like "wings" around, described by the regime $a_0^2 < a_1^2$ and $a_0^2 < a_2^2$, while, for example, in [9], there were considered mainly time-like events inside the light bipyramid $(a_0^2 > a_1^2 \text{ and } a_0^2 > a_2^2)$. Time-like region is marked as dark at Fig. 1.

We underline that the existence of such wings has been overlooked in the paper [4], what has its consequences in possible physical interpretation proposed in next sections. This seems to reveal some new aspects of causality in scator-deformed Lorentz metric (see the picture at end of the paper).

A closely related notion, "super super-restricted space conditions", was introduced in [9] without a direct relation to the type of considered events. Super-restricted space conditions define either time-like events (in even-dimensional spaces) or space-like events (in odd-dimensional spaces).

The paper is organized as follows. In section 2 we introduce basic objects and transformations responsible for isometries in the scator space S. Then, in sections 3 and 4 we propose and develop a new framework in which calculation proceed in a more natural way, using a distributive product. In section 5 we continue along these lines focusing on isometries. Section 6 is entirely devoted to the question of metric properties of scators; in particular, we obtain possible closest analogue of scalar product we can get, although it is even not bilinear. The last section contains physical comments and conclusions.

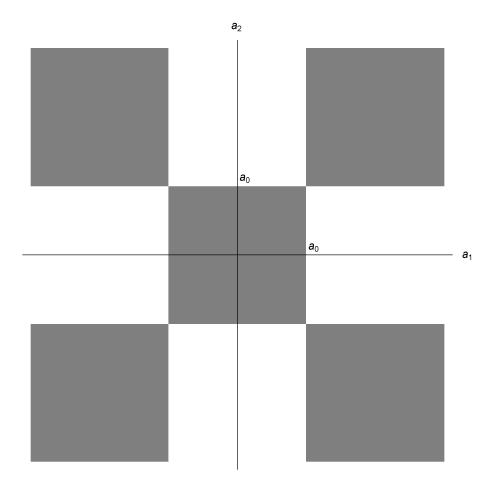


Figure 1: Time-like wings at a fixed time a_0 ($a_0 \neq 0$). Duality operations (section 2) take us out from the bipyramid (represented in this cross-section by the inner square) to other causal realms, and conversely.

2 Dualities - phenomenological treatment

Now we turn our attention to the issue of isometries in the scator space S. We begin with defining some operations and then check their properties.

Definition 2.1. If we have a 3-scator $\overset{o}{a}=(a_0;a_1,a_2)$, then we call the scator of the form

$$\frac{\overset{o}{a}}{=} \left(\frac{a_1 a_2}{a_0}, a_2, a_1\right) \tag{2.1}$$

its dual (or ordinary dual) scator, and star denotes hypercomplex conjugate:

$$\stackrel{o}{a^*} = (a_0; -a_1, -a_2). \tag{2.2}$$

Lemma 2.2. Operation of duality commutes with hypercomplex conjugation.

Proof: We have

$$(\stackrel{o}{\bar{a}})^* = \left(\frac{a_1 a_2}{a_0}, a_2, a_1\right)^* = \left(\frac{a_1 a_2}{a_0}, -a_2, -a_1\right)$$
 (2.3)

and

$$\frac{o}{(a^*)} = \overline{(a_0; -a_1, -a_2)} = \left(\frac{a_1 a_2}{a_0}, -a_2, -a_1\right)$$
(2.4)

which are evidently equal.

Lemma 2.3. Operation of duality is idempotent: $\frac{\circ}{(\bar{a})} = \overset{\circ}{a}$.

Proof: This follows instantly by straightforward calculation. \Box

Remark 2.4. Hypercomplex conjugation is a homomorphism in S, i.e.,

$$(\stackrel{\circ}{ab})^* = (\stackrel{\circ}{a})^*(\stackrel{\circ}{b})^*$$
 (2.5)

Operation of duality does not provide yet another homomorphic sctructure in S, i.e. $\frac{\circ}{ab} \neq \frac{\circ}{a}\frac{\circ}{b}$.

The above statements follow from considerations similar to direct calculations included in [7]. Soon we will get a better understanding of these facts by applying a new approach which is both faster and simpler.

Proposition 2.5. Operation of duality preserves the scator product:

$$\frac{\overset{o}{a}\overset{o}{b}}{=}\overset{o}{a}\overset{o}{b}.\tag{2.6}$$

Proof: We present explicit calculation for the scalar component:

$$(\overset{\circ}{ab})_0 = a_0b_0 + a_1b_1 + a_2b_2 + \frac{a_1a_2b_1b_2}{a_0b_0},$$
 (2.7)

and

$$(\bar{a}\bar{b})_{0} = \bar{a}_{0}\bar{b}_{0} + \bar{a}_{1}\bar{b}_{1} + \bar{a}_{2}\bar{b}_{2} + \frac{\bar{a}_{1}\bar{a}_{2}\bar{b}_{1}\bar{b}_{2}}{\bar{a}_{0}\bar{b}_{0}} =
 = \frac{a_{1}a_{2}b_{1}b_{2}}{a_{0}b_{0}} + a_{2}b_{2} + a_{1}b_{1} + \frac{a_{1}a_{2}b_{1}b_{2}}{\frac{a_{1}a_{2}b_{1}b_{2}}{a_{0}b_{0}}} =
 = a_{0}b_{0} + a_{1}b_{1} + a_{2}b_{2} + \frac{a_{1}a_{2}b_{1}b_{2}}{a_{0}b_{0}} = (\tilde{a}\tilde{b})_{0}.$$
(2.8)

Similar direct computation can be done for director components. \Box

Proposition 2.6. Operation of duality is an isometry in 3-scator space.

Proof: For ordinary scator $\overset{o}{a}$ we have its norm

$$\|\overset{o}{a}\|^2 = \overset{o}{a}\overset{o}{a}^* = a_0^2 \left(1 - \frac{a_1^2}{a_0^2}\right) \left(1 - \frac{a_2^2}{a_0^2}\right). \tag{2.9}$$

Thus, for the dual scator $\frac{o}{a}$, we get

$$\begin{split} \|\bar{a}\|^2 &= \frac{\overset{o}{a}\overset{o}{a}^*}{\bar{a}} = \left(\frac{a_1 a_2}{a_0}, a_2, a_1\right) \left(\frac{a_1 a_2}{a_0}, -a_2, -a_1\right) = \\ &= \frac{a_1^2 a_2^2}{a_0^2} \left(1 - \frac{a_2^2}{\frac{a_1^2 a_2^2}{a_0^2}}\right) \left(1 - \frac{a_1^2}{\frac{a_1^2 a_2^2}{a_0^2}}\right) = \\ &= \frac{a_1^2 a_2^2}{a_0^2} + a_0^2 - a_1^2 - a_2^2 = a_0^2 \left(1 - \frac{a_1^2}{a_0^2}\right) \left(1 - \frac{a_2^2}{a_0^2}\right), \end{split}$$
(2.10)

which exactly coincides with the norm of the original scator. \Box

Remark 2.7. Taking into account (1.1), (2.2) and (2.1), we can easily verify that

$$a^{\circ}b + a^{\circ}b^{\circ} = 2\left(a_0b_0 - a_1b_1 - a_2b_2 + \frac{a_1a_2b_1b_2}{a_0b_0}\right),\tag{2.11}$$

where the right-hand side is proportional to 1 (omitted for simplicity here and in many other places).

Definition 2.8. If we have a 3-scator $\overset{\circ}{a} = (a_0; a_1, a_2)$, then we call a scator of the form

$$\frac{o}{\bar{a}_i} = \left(a_1; a_0, \frac{a_1 a_2}{a_0} \right)$$
(2.12)

its internal dual scator, and a scator of the form

$$\frac{o}{\bar{a}_e} = \left(a_2; \frac{a_1 a_2}{a_0}, a_0 \right)$$
(2.13)

its external dual.

Lemma 2.9. Internal and external duality operations anti-commute with hypercomplex conjugation.

Proof: We have

$$(\bar{a}_i)^* = \left(a_1; a_0, \frac{a_1 a_2}{a_0}\right)^* = \left(a_1; -a_0, -\frac{a_1 a_2}{a_0}\right)$$
 (2.14)

and

$$\frac{o}{(a^*)_i} = \overline{(a_0; -a_1, -a_2)_i} = \left(-a_1; a_0, \frac{a_1 a_2}{a_0}\right),$$
(2.15)

so that

$$\left(\stackrel{o}{\bar{a}}\right)_{i}^{*} + \overline{\left(a^{*}\right)_{i}} = 0. \tag{2.16}$$

Similarly we have

$$(\stackrel{o}{\bar{a}})_e^* = (a_2; \frac{a_1 a_2}{a_0}, a_0)^* = \left(a_2; -\frac{a_1 a_2}{a_0}, -a_0\right)$$
 (2.17)

and

$$\frac{o}{(a^*)_e} = \overline{(a_0; -a_1, -a_2)_e} = \left(-a_2; \frac{a_1 a_2}{a_0}, a_0\right), \tag{2.18}$$

so that

$$(\stackrel{o}{\bar{a}}_e)^* + \overline{(a^*)_e} = 0,$$
 (2.19)

which ends the proof.

Lemma 2.10. Internal and external duality operations are idempotent.

Proof: It is enough to apply twice definitions of both operations. \Box

Remark 2.11. Operation of external and internal duality do not provide yet another homomorphic sctructures in S, so i.e.

$$(\overline{ab})_i \neq \overset{o}{\bar{a}}_i \overset{o}{\bar{b}}_i, \qquad (\overset{o}{ab})_e \neq \overset{o}{\bar{a}}_e \overset{o}{\bar{b}}_e,$$
 (2.20)

which follows from straightforwad calculation.

Lemma 2.12. Operations of internal and external duality preserve scator product.

Proof: By direct computation, similar to the case of ordinary duality. \Box

Definition 2.13. A transformation that exchanges time-like events with space-like events (and conversely) and leaves the type of light-like events unchanged is called a causality swap (or a pseudo-isometry).

Proposition 2.14. Both internal and external duality operations are causality swaps of 3-scator space.

Proof: Denoting $\overset{o}{\bar{a}}_i = (\bar{a}_{0i}; \bar{a}_{1i}, \bar{a}_{2i})$, we compute:

$$\begin{split} \|\overset{o}{\bar{a}}_{i}\|^{2} &= \bar{a}_{0i}^{2} \left(1 - \frac{\bar{a}_{1i}^{2}}{\bar{a}_{0i}^{2}} \right) \left(1 - \frac{\bar{a}_{2i}^{2}}{\bar{a}_{0i}^{2}} \right) = \\ &= \bar{a}_{0i}^{2} \left(1 - \frac{\bar{a}_{1i}^{2}}{\bar{a}_{0i}^{2}} - \frac{\bar{a}_{2i}^{2}}{\bar{a}_{0i}^{2}} + \frac{\bar{a}_{1i}^{2}}{\bar{a}_{0i}^{2}} \frac{\bar{a}_{2i}^{2}}{\bar{a}_{0i}^{2}} \right) = a_{1}^{2} \left(1 - \frac{a_{0}^{2}}{\bar{a}_{1}^{2}} \right) \left(1 - \frac{a_{2}^{2}}{\bar{a}_{1}^{2}} \right) = \\ &= -a_{0}^{2} \left(1 - \frac{a_{1}^{2}}{\bar{a}_{0}^{2}} \right) \left(1 - \frac{a_{2}^{2}}{\bar{a}_{0}^{2}} \right) = -\|\overset{o}{a}\|^{2} \end{split}$$

$$(2.21)$$

Therefore, if original scator represents a time-like event, then its internal dual has to represent a space-like event, and $vice\ versa$. Light-like scators do not change their type. Analogous computation, with the same consequences, can be done for the external dual.

Corollary 2.15. Duality operations commuting with hypercomplex conjugation are isometries, while duality operations anti-commuting with hypercomplex conjugation are causality swaps.

Finally, we arrive at a very strong theorem providing some kind of translator between different kinds of duals.